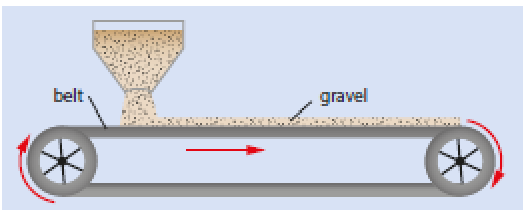
Teacher notes
Topic A

A constant net force does not necessarily imply acceleration when the mass is changing.

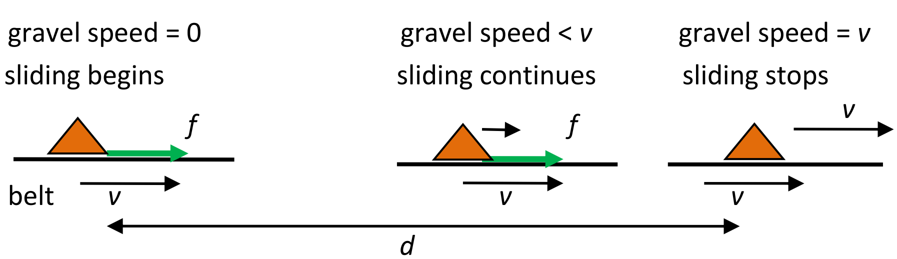
The figure shows a typical example of a system with changing mass:



The gravel falling on the belt increases the mass on the belt. Gravel falls on the belt at a rate of μ kg per second. The belt is moving at constant speed v. The acceleration is zero so we would expect a zero force acting on the belt. But this is not correct because the mass is changing. We need to use $F_{net} = \frac{\Delta p}{\Delta t}$ and apply it to the system of the belt and all the gravel (including that in the hopper). At $t = 0$ the momentum of the belt and the gravel on it is $p = Mv$. After time Δt, the mass on the belt will increase by $\Delta m = \mu \Delta t$ and so the momentum will be $p' = (M + \Delta m)v = (M + \mu \Delta t)v$. The *change* in momentum is thus $\Delta p = v\mu\Delta t$ and so the force is $F_{net} = \frac{v\mu\Delta t}{\Delta t} = \mu v$. This is a force exerted on the belt by the motor turning the belt. We have a net force but no acceleration.

The power provided by the motor is $P = Fv = \mu v^2$. But the kinetic energy of the gravel falling on the belt is increasing at a rate: $\frac{1}{2}\frac{\Delta m}{\Delta t}v^2 = \frac{1}{2}\mu v^2$. Where did the other $\frac{1}{2}\mu v^2$ go?

When the mass Δm falls on the belt its horizontal velocity is zero and so will slide on the belt for a distance d until a frictional force f accelerates the mass to the speed of the belt.

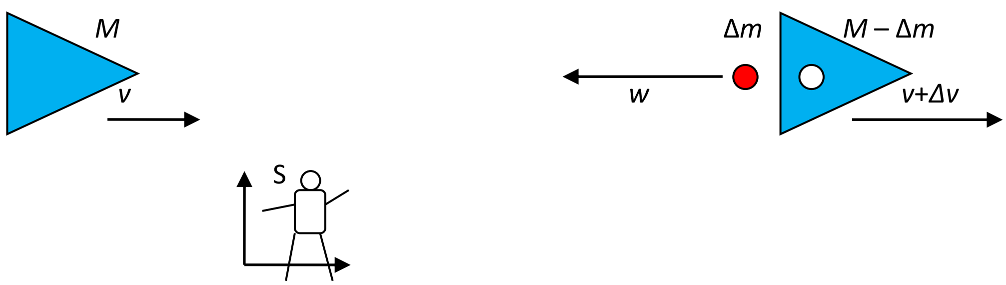


The work done by the frictional force is *fd*.

The acceleration of the mass is $a=\frac{f}{\Delta m}$ and $v^2=2ad=2\frac{f}{\Delta m}d$ so that $\frac{1}{2}\Delta m v^2=fd$. The rate at which the frictional force does work is thus $\frac{1}{2}\mu v^2$ and represents the rate at which energy is lost to friction, i.e. heat.

Thus, half the power provided by the motor is wasted as heat as the gravel accelerates from zero to *v* by sliding over the belt for some distance.

And now to rockets! We showed in the textbook that the rocket equation is $M\frac{\Delta v}{\Delta t}=\mu u$ where *M* is the mass of the rocket (including fuel) and *u* the exhaust speed of the gases *relative to the rocket*. It is instructive to re-derive this formula. The figure shows the burned fuel being ejected with speed *w* relative to some stationary observer S. All speeds in the figure are relative to this observer.



The change in momentum of the rocket-gas system is then

$$\begin{aligned}\Delta p &= (M-\Delta m)(v+\Delta v)-w\Delta m - Mv\\ &= Mv + M\Delta v - v\Delta m - \Delta m\Delta v - w\Delta m - Mv\\ &= M\Delta v - v\Delta m - w\Delta m - \Delta m\Delta v\\ &= M\Delta v-(v+w+\Delta v)\Delta m\end{aligned}$$

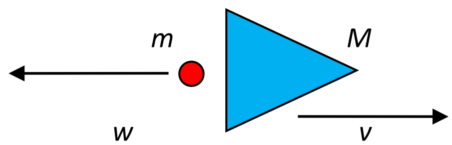
But $w+v+\Delta v$ is the relative speed of the exhaust gases *with respect to the rocket* $(w-(-v-\Delta v))$, i.e. $u=w+v+\Delta v$ and so

$$\Delta p = M\Delta v - u\Delta m.$$

Since there are no external forces, $\Delta p=0$ and so $M\Delta v=u\Delta m$ or $M\frac{\Delta v}{\Delta t}=u\frac{\Delta m}{\Delta t}$ i.e. $M\frac{\Delta v}{\Delta t}=\mu u$.

It is interesting to also find the force that actually accelerates the rocket (the thrust): the change in momentum of the burned fuel is $-w\Delta m-(v\Delta m)=-(w+v)\Delta m=-u\Delta m$. The magnitude of the rate of change of this momentum is $u\frac{\Delta m}{\Delta t}=\mu u$. This is the force on the fuel and so by Newton's third law this is also the force on the rocket (the thrust).

The rocket formula can also be derived by applying $F_{ext}=\frac{\Delta p}{\Delta t}$ directly to the system consisting of the rocket and the ejected fuel: The figure shows the rocket when its mass is *M* and the ejected fuel of mass *m*.



The external force on the system of rocket + fuel is zero and so $0=\frac{\Delta p}{\Delta t}$ where *p* is the momentum of the system: $p=Mv-mw$. Then $\frac{\Delta p}{\Delta t}=\frac{\Delta M}{\Delta t}v+M\frac{\Delta v}{\Delta t}-\frac{\Delta m}{\Delta t}w-m\frac{\Delta w}{\Delta t}$. But $\frac{\Delta w}{\Delta t}=0$ (no forces act on the fuel once out of the rocket) and $\frac{\Delta M}{\Delta t}=-\frac{\Delta m}{\Delta t}=-\mu$ so that $0=-\mu(v+w)+M\frac{\Delta v}{\Delta t}$ i.e. $M\frac{\Delta v}{\Delta t}=\mu(v+w)$. But $v+u$ is the speed *u* of the fuel relative to the rocket and so

$$M\frac{\Delta v}{\Delta t}=\mu u$$

It is important to notice that if we apply $F_{ext}=\frac{\Delta p}{\Delta t}$ to *just the rocket* we will get the wrong answer:

$F_{ext}=\frac{\Delta p}{\Delta t}=\frac{\Delta(Mv)}{\Delta t}=\frac{\Delta M}{\Delta t}v+M\frac{\Delta v}{\Delta t}=-\mu v+M\frac{\Delta v}{\Delta t}$. With $F_{ext}=0$ we get the wrong result $M\frac{\Delta v}{\Delta t}=\mu v$.

This is because $F_{ext}=\frac{\Delta p}{\Delta t}$ is valid for a system whose mass stays constant (this still allows for mass from one part of the system to move to another part but the mass of the system as a whole must stay the same). Thus we can apply $F_{ext}=\frac{\Delta p}{\Delta t}$ to the system of rocket + fuel (as we did above) but not to the rocket alone. This is also why we got the correct answer when applied it to the system of the gravel on the belt and in the hopper in the first example in this note.

A rocket of total mass (including fuel) M_0 = 3.0×10^4 kg is in outer space (no gravity). The rocket ejects burned fuel at a rate of μ = 2.0×10^2 kg s^{-1} with speed u = 3.0×10^2 m s^{-1} (relative to the rocket).

(a) What is the force accelerating the rocket?
(b) What is the acceleration of the rocket at t = 0 and at t = 10 s?
(c) The mass of the fuel is 80% of the total initial mass of the rocket. When does the fuel run out?
(d) What is the acceleration of the rocket the instant the fuel runs out?

Answers

(a) The force on the ejected fuel is the rate of change of its momentum i.e. $\mu u = 200\times300 = 6.0\times10^4$ N. By Newton's third law this is the force accelerating the rocket.
(b) After time t the mass of the rocket is $M = M_0 - \mu t$. So at t = 0, from the rocket equation, $a=\frac{F}{M}=\frac{6.0\times10^4}{3.0\times10^4}=2.0\text{ m s}^{-2}$. At t = 10 s, $a=\frac{F}{m}=\frac{6.0\times10^4}{3.0\times10^4-3.0\times10^2\times10}=2.2\text{ m s}^{-2}$.
(c) The mass of the fuel is $0.80\times3.0\times10^4 = 2.4\times10^4$ kg. So it runs out after $\frac{2.4\times10^4}{300}=80\text{ s}$.
(d) The acceleration will be $a=\frac{F}{m}=\frac{6.0\times10^4}{3.0\times10^4-3.0\times10^2\times80}=10\text{ m s}^{-2}$.

Now suppose that the rocket of the previous problem was on the surface of a moon where the acceleration of gravity is 3.0 m s^{-2}. The engines are turned on. Will the rocket leave the surface of the moon?

The thrust is less than the weight so the rocket cannot move right away. We must wait until the weight of the rocket is reduced to just below the thrust and then we will have lift off. Thus the weight has to be reduced to 6.0×10^4 N and so the mass will be 2.0×10^4 kg. The rocket must burn 1000 kg of fuel and this takes 3.3 s; this is when the rocket will begin to move upwards.